MA 362 final exam review problems

Hopefully final version as of May 1st

- The final will be on Friday, May 5th, from 8:00 to 10:00 am, in ME 1061.
- It will cover all the material we have done since Homework 5.
- Most of the problems on the exam will be closely based on ones from the list below, and on ones from Midterm 2 and its review problems.
- For each problem, you must explain your reasoning.
- Note that these are not arranged in order of difficulty!

1. Sketch the region given by the inequalities $x^2+y^2+z^2 \leq 4$, $x^2+y^2 \leq 1$, and find its volume and surface area.

2. Evaluate $\iint_S dx \wedge dy$, where S is the surface $x^2+y^2+z^2=3$ with $x \leq 0$, $y \geq 0$, and $z \leq 0$, oriented by a normal vector pointing toward the origin.

3. Evaluate $\iint_S xy dS$, where S is the part of the surface $z = x^2+y^2$ given by $z \leq 1$. Evaluate $\iint_S \cos(z^3) dy \wedge dz + e^{x^2 z^2} dz \wedge dx + z dx \wedge dy$ for the same surface, oriented by the normal pointing upwards.

4. Let C be the curve given by $y = z = \sqrt{1-x^2-y^2}$.
   (a) Sketch C, and find its arc length.

   (b) Mark an orientation for C on your sketch (it can be any orientation you like), and evaluate $\int_C e^{y^2-z^2} dx + 2xye^{y^2-z^2} dy - 2xze^{y^2-z^2} dz$ for that orientation.

5. Evaluate $\int_C (z^2+yz\sin(xyz))dx+(y^2+xz\sin(xyz))dy+(x+xy\sin(xyz))dz$ where C is the curve following the outline for the triangle from $(1,0,0)$ to $(0,1,0)$ to $(0,0,1)$ and back to $(1,0,0)$.

6. Let a and b be real numbers such that $0 < a < b$. What is the flux of the vector field $(x,y,z)/(x^2+y^2+z^2)^{3/2}$ outward through the boundary of the region $a^2 \leq x^2+y^2+z^2 \leq b^2$? What is the flux outward through the boundary of the region $x^2+y^2+z^2 \leq a^2$?

7. Evaluate $\int_C \left(\cos(x+y+z)+x^2\right)(dx+dy+dz)$, where C is the curve following the outline of the parallelogram from $(1,2,3)$ to $(0,4,2)$ to $(2,5,2)$ to $(3,3,3)$ and back to $(1,2,3)$.

8. Which of the following differential two forms can be written as $d\omega$ for some one form ω? If the answer is yes, find such a one form ω.

   (a) $-xdy \wedge dz + (x - z)dz \wedge dx + zdx \wedge dy$,

   (b) $xdy \wedge dz + (x - z)dz \wedge dx + zdx \wedge dy$,

   (c) $2yzdy \wedge dz + 3x^2zdz \wedge dx + xdx \wedge dy$.

9. Find the flux of the vector field $(0, 0, \sin^2(x^2+y^2)+z)$ through the surface given by $z = \cos^2(x^2 + y^2) + e^{x^2+y^2}$ and $x^2 + y^2 \leq 1$, oriented upward.

10. Evaluate
$$\iint_S y^2zdy \wedge dz + (x + 1)^z dz \wedge dx,$$
where S is the surface given by $x = y^2$, $0 \leq z \leq 3$, $x \leq 8$, oriented towards the x axis.

Formula sheet

- The arc length of the path $(x(t), y(t))$ from t_0 to t_1 is $\int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2} dt$.

- Polar, cylindrical, and spherical coordinates are given by $x = r\cos\theta = \rho\sin\varphi\cos\theta$, $y = r\sin\theta = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$. Moreover $dxdydz = rdrd\theta dz = \rho^2 \sin\varphi d\rho d\theta d\varphi$.

- Integral formulas:

$$\int_C F_1 dx + F_2 dy = \int_a^b F_1(x(t), y(t))x'(t)dt + F_2(x(t), y(t))y'(t)dt,$$

  where $(x(t), y(t))$, $a \le t \le b$ is a parametrization of C.

$$\int_C \partial_x f dx + \partial_y f dy = f(q) - f(p),$$

  where C is a curve from p to q.

$$\iint_D (\partial_x F_2 - \partial_y F_1) dxdy = \int_{\partial D} F_1 dx + F_2 dy,$$

  where ∂D is the boundary of D oriented so that D is to the left.
  If $x = x(u, v)$ and $y = y(u, v)$, then

$$\iint_D f dxdy = \iint_{D^*} f \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv,$$

  where $\partial(x, y)/\partial(u, v) = \partial_u x \partial_v y - \partial_v x \partial_u y$ is the determinant of the Jacobian matrix. Here D is a region in the xy plane, and D^* is the corresponding region in the uv plane.

- If $f = f(x, y)$, then $df = \partial_x f dx + \partial_y f dy$.

- If $\alpha = F_1 dx + F_2 dy$, then $d\alpha = (\partial_x F_2 - \partial_y F_1) dx \wedge dy$ and $*\alpha = -F_2 dx + F_1 dy$.
  If further $\beta = G_1 dx + G_2 dy$, then $\alpha \wedge \beta = (F_1 G_2 - F_2 G_1) dx \wedge dy$.

- If $f = f(x, y, z)$, then $df = \partial_x f dx + \partial_y f dy + \partial_z f dz$.

- If $\alpha = F_1 dx + F_2 dy + F_3 dz$, then

  $d\alpha = (\partial_y F_3 - \partial_z F_2) dy \wedge dz + (\partial_z F_1 - \partial_x F_3) dz \wedge dx + (\partial_x F_2 - \partial_y F_1) dx \wedge dy.$

  If further $\beta = G_1 dx + G_2 dy + G_3 dz$ then

  $\alpha \wedge \beta = (F_2 G_3 - F_3 G_2) dy \wedge dz + (F_3 G_1 - F_1 G_3) dz \wedge dx + (F_1 G_2 - F_2 G_1) dx \wedge dy.$

If further $\gamma = H_1 dy \wedge dz + H_2 dz \wedge dx + H_3 dx \wedge dy$, then

$$\alpha \wedge \gamma = (F_1 H_1 + F_2 H_2 + F_3 H_3) dx \wedge dy \wedge dz,$$

and

$$d\gamma = (\partial_x H_1 + \partial_y H_2 + \partial_z H_3) dx \wedge dy \wedge dz.$$

- More integral formulas:

$$\iint_S F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy = \iint_D (F_1, F_2, F_3) \cdot (T_u \times T_v) du dv,$$

where $(x(u,v), y(u,v), z(u,v))$ with u and v in D is a correctly oriented parametrization of S, and $T_u = \partial_u(x, y, z)$ and $T_v = \partial_v(x, y, z)$. Also

$$\iint_D \|T_u \times T_v\| du dv.$$

gives the area of S.

$$\iint_S d\omega = \int_{\partial S} \omega,$$

where ω is a one form and S is a surface with ∂S oriented so that S is to the left.

$$\iiint_V d\omega = \iint_{\partial V} \omega,$$

where ω is a two-form and V is a region with ∂V oriented outward.