MA 362 second midterm review problems

Hopefully final version as of March 28th

- The second midterm will be on Monday, April 3rd, from 8:00 to 9:30 pm, in MTHW 210.
- It will cover all the material we have done since Homework 5.
- Most of the problems on the exam will be closely based on ones from the finalized version of the list below (but the actual exam will be much shorter).
- For each problem, you must explain your reasoning.
- Note that these are not arranged in order of difficulty!

1. Which of the following differential 1 forms can be written as df for some function $f\colon \mathbb{R}^2 \to \mathbb{R}$? If the answer is yes, find such a function f.

   (a) $(3 + 2xy)dx + (x^2 - 3y^2)dy$,

   (b) $(2x\cos y - y\cos x)dx + (-x^2 \sin y - \sin x)dy$,

   (c) $(ye^x + \sin y)dx + (e^x + x\cos y)dy$

2. Evaluate
$$\int_C \sqrt{e^{\cos^3(x+y)}}(dx + dy),$$
where C is the circular arc beginning at $(0, 0)$, passing through $(2, 8)$, and ending at $(1, -1)$. Evaluate the same integral where C is a curve beginning at $(2\pi, -\pi)$ and ending at $(-\pi, 2\pi)$

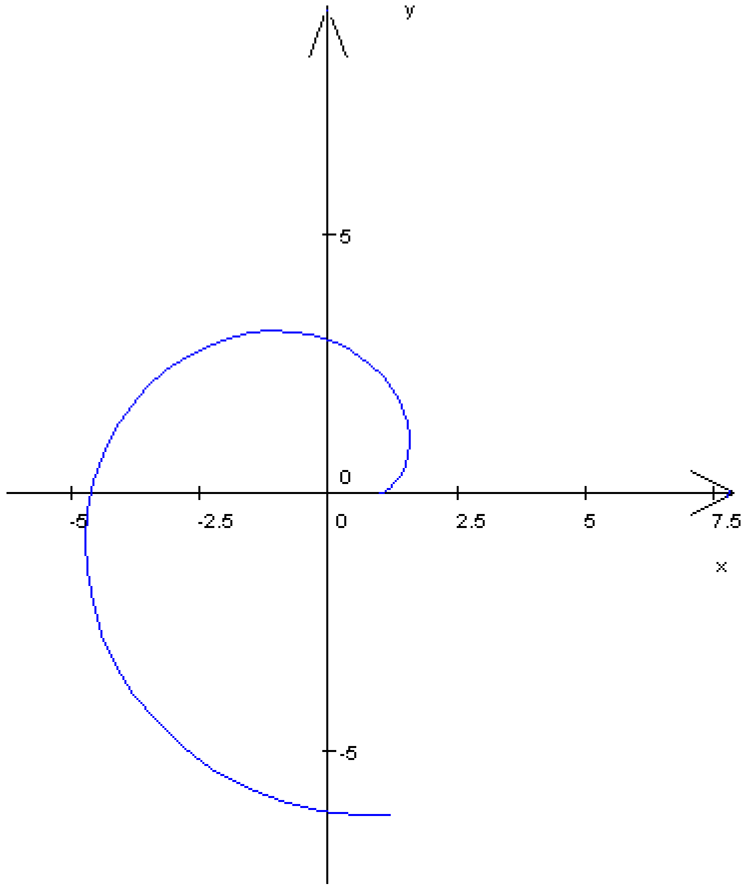
3. Evaluate
$$\int_C \frac{(x + y)dx + (y - x)dy}{x^2 + y^2},$$
where

   (a) C is the circle $x^2 + y^2 = R^2$ for some $R > 0$, oriented counterclockwise;

   (b) C is the circle $(x - 3)^2 + (y - 4)^2 = R^2$ for some $R > 0$ with $R \neq 5$, oriented clockwise

4. A string of length 2π starts out wound around a spool given by the unit

circle in $\mathbb{R}^2$, beginning and ending at $(1,0)$. The top end of the string is unwound, while being kept taut, until it reaches the point $(1,-2\pi)$; the curve C traced by this end is called an *involute* of the circle. Here is a picture of C:



(a) Find the length of C.

(b) Find the area of the region bounded by C and the straight line segment joining its endpoints.

Hint: C is parametrized by $x(t) = \cos t + t\sin t$ and $y(t) = \sin t - t\cos t$.

5. Let $f(x,y) = 3x^2y - y^3$. Find df, $*df$, ddf, $d*df$, $df \wedge df$, and $df \wedge *df$.

6. Which of the following differential 1 forms can be written as df for some function $f\colon \mathbb{R}^3 \to \mathbb{R}$? If the answer is yes, find such a function f.

(a) $xdx + ydy + zdz$

(b) $xy^2z^3dx + 2x^2yz^3dy + 3x^2y^2z^2dz$

(c) $e^x dx + e^z dy + e^y dz$

(d) $yze^{xz} dx + e^{xz} dy + xye^{xz} dz$

7. Sketch the curve given by the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $x + y + z = 1$. Find a constant a such that the arclength of this curve can be written as

$$a \int_0^{2\pi} \sqrt{1 - \cos t \sin t} dt.$$

8. Let $c(t) = (4 \sin t, 5 \cos t, 3 \sin t)$. Find $\|c(t)\|$ and $\|c'(t)\|$. Find a plane containing the curve traced out by $c(t)$, and sketch the curve. Which coordinate axes and coordinate planes does the curve cross, and where does it cross them? Find the arc length of the curve.

9. (a) Evaluate

$$\int_C yze^{xz} dx + e^{xz} dy + xye^{xz} dz,$$

where C is the parametric curve (t, t^2, t^4) with $0 \leq t \leq 1$.

(b) Find an oriented curve C such that

$$\int_C yze^{xz} dx + e^{xz} dy + xye^{xz} dz = 1$$

Hint: Look at problem 6d above.

10. Evaluate

$$\int_C xy^2 dx - x^2 y dy,$$

where C is the oriented curve consisting of the line segment from $(-2, 0)$ to $(2, 0)$ followed by the top half of the circle $x^2 + y^2 = 4$ from $(2, 0)$ back to $(-2, 0)$.

11. Let $R > 0$. Evaluate

$$\iint_D e^{x^2+y^2} dx dy,$$

(a) where D is the region given by $x^2 + y^2 \leq R^2$,

(b) where D is the region given by $x^2 + y^2 \leq R^2$ and $x + y \geq 0$,

(c) where D is the region given by $x^2 + y^2 \leq R^2$ and $x - y \leq 0$,

12. Let D be the region given by $x^2 + y^2 \leq 1$, $(x-1)^2 + y^2 \geq 1$, $x \geq 0$, $y \geq 0$.
    (a) Sketch D.

    (b) Let $u = x^2 + y^2$, $v = x^2 + y^2 - 2x$, and let D^* be the region in the uv plane corresponding to D. Sketch D^*

    (c) Evaluate $\iint_D xydxdy$.

13. Let a be a constant. For which values of a is it possible to find F_1, F_2, F_3 such that

$$d(F_1dx + F_2dy + F_3dz) = (ax+1)dy \wedge dz + 2dz \wedge dx + 3dx \wedge dy?$$

For such a, give an example of F_1, F_2, and F_3 that work.

Formula sheet

- The arc length of the path $(x(t), y(t))$ from t_0 to t_1 is

$$\int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2} dt$$

- Polar coordinates are given by $x = r\cos\theta$, $y = r\sin\theta$. Moreover $dxdy = rdrd\theta$.

- Integral formulas:

$$\int_C F_1 dx + F_2 dy = \int_a^b F_1(x(t), y(t))x'(t)dt + F_2(x(t), y(t))y'(t)dt,$$

  where $(x(t), y(t))$, $a \le t \le b$ is a parametrization of C.

$$\int_C \partial_x f dx + \partial_y f dy = f(q) - f(p),$$

  where C is a curve from p to q.

$$\iint_D (\partial_x F_2 - \partial_y F_1) dxdy = \int_{\partial D} F_1 dx + F_2 dy,$$

  where ∂D is the boundary of D oriented so that D is to the left.
  If $x = x(u, v)$ and $y = y(u, v)$, then

$$\iint_D f dxdy = \iint_{D^*} f \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv,$$

  where $\partial(x, y)/\partial(u, v) = \partial_u x \partial_v y - \partial_v x \partial_u y$ is the determinant of the Jacobian matrix. Here D is a region in the xy plane, and D^* is the corresponding region in the uv plane.

- If $f = f(x, y)$, then $df = \partial_x f dx + \partial_y f dy$.

- If $\alpha = F_1 dx + F_2 dy$, then $d\alpha = (\partial_x F_2 - \partial_y F_1) dx \wedge dy$ and $*\alpha = -F_2 dx + F_1 dy$.
  If further $\beta = G_1 dx + G_2 dy$, then $\alpha \wedge \beta = (F_1 G_2 - F_2 G_1) dx \wedge dy$.

- If $f = f(x, y, z)$, then $df = \partial_x f dx + \partial_y f dy + \partial_z f dz$.

- If $\alpha = F_1 dx + F_2 dy + F_3 dz$, then

$$d\alpha = (\partial_y F_3 - \partial_z F_2) dy \wedge dz + (\partial_z F_1 - \partial_x F_3) dz \wedge dx + (\partial_x F_2 - \partial_y F_1) dx \wedge dy.$$

  If further $\beta = G_1 dx + G_2 dy + G_3 dz$ then

$$\alpha \wedge \beta = (F_2 G_3 - F_3 G_2) dy \wedge dz + (F_3 G_1 - F_1 G_3) dz \wedge dx + (F_1 G_2 - F_2 G_1) dx \wedge dy.$$

  If further $\gamma = H_1 dy \wedge dz + H_2 dz \wedge dx + H_3 dx \wedge dy$, then

$$\alpha \wedge \gamma = (F_1 H_1 + F_2 H_2 + F_3 H_3) dx \wedge dy \wedge dz,$$

  and

$$d\gamma = (\partial_x H_1 + \partial_y H_2 + \partial_z H_3) dx \wedge dy \wedge dz.$$