

1. Let $f(X) \in \mathbb{Q}[X]$ be an irreducible polynomial of degree $n > 2$ such that $f(X) = f(-X)$. Prove that the galois group of f is not the symmetric group S_n .
2. Let K be the splitting field over $\mathbb{Q}$ of $X^4 - 2X^2 - 1$.
 - (a) Determine the galois group of $K/\mathbb{Q}$.
 - (b) Show that the only three subfields of K having degree 2 over $\mathbb{Q}$ are $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(\sqrt{-2})$.
3. D&F, p. 582, #11.
4. Let k be a field of characteristic $\neq 2$, and let $f \in k[X]$ be an irreducible polynomial of degree 4. If r_1, r_2, r_3 and r_4 are the roots of f (in some splitting field), then the polynomial g whose roots are $r_1r_2 + r_3r_4$, $r_1r_3 + r_2r_4$ and $r_1r_4 + r_2r_3$ is called the *resolvent cubic* of f .
 - (a) Show that the discriminant of f is the same as that of g .
 - (b) Let $G \subset S_4$ be the galois group of f , and let $V \triangleleft S_4$ be the unique normal subgroup of order 4. Prove that the fixed field T of $V \cap G$ is a splitting field of g .
 - (c) Let $t = [T : k]$ (see (b)). Prove that $G = S_4, A_4$ or V according as $t = 6, 3$, or 1. What are the possibilities for G when $t = 2$?
 - (d) Can the roots of $X^4 + X - 5 \in \mathbb{Q}[X]$ be constructed with ruler and compass? (Hint. Compute the resolvent cubic—see Homework 1.)
 - (e) Determine the galois group for the minimal polynomial over $\mathbb{Q}$ of each of

$$\sqrt{3 + 2\sqrt{2}}, \quad \sqrt{7 + 2\sqrt{10}}, \quad \sqrt{5 + 2\sqrt{5}}, \quad \sqrt{5 + 2\sqrt{21}}.$$
 (Cf. D&F, p. 618, #13.)