

1. Using D&F, p.254, Prop.11, show that a commutative ring that has no maximal ideals must be the trivial ring (i.e., it has just one element.)

2. Show that the intersection of all the prime ideals in a commutative ring R is the ideal

$$N := \{x \in R \mid x^n = 0 \text{ for some } n > 0\}.$$

(This N is called the *nilradical* of R ; and elements of N are called *nilpotent*.)

Hint: Start by showing that for any $x \in N$, the localization R_x has no prime ideals.

3. Show that the intersection of all the prime ideals containing a given ideal I in a commutative ring R is the ideal

$$\sqrt{I} := \{x \in R \mid x^n \in I \text{ for some } n > 0\}.$$

(This ideal, called the *radical of I* , is denoted $\text{rad } I$ in D&F.)

Hint: Consider R/I .

EXTRA CREDIT

4. Show that in Theorem 32 on page 700 of D&F, $\mathcal{I}(\mathcal{Z}(I))$ is the intersection of all the *maximal* ideals of $k[x_1, x_2, \dots, x_n]$ containing I .